Inverse dynamics for a three-link planar chain


The following convention applies to the notation used in this paper:

Subscript \(i\) runs 1, 2, or 3 meaning shoulder, elbow, or wrist joint when referring to angles, joint moments, or joint reaction forces; or meaning upper arm, forearm, or hand segment when referring to everything else.

\(x_i, y_i\) refer to the position of the center of mass of segment \(i\) in the horizontal or vertical direction, respectively.

\(l_i\) is the length of segment \(i\)

\(d_i\) is the distance from the proximal joint of the segment \(i\) to its center of mass position.

\(d_S\) is the distance from the wrist joint to the point of application of the spring force.

\(m_i\) is the mass of segment \(i\)

\(I_i\) is the moment of inertia of segment \(i\)

\(F_{xi}, F_{yi}\) are the joint reaction forces of joint \(i\) in the horizontal or vertical direction, respectively.

\(F_S\) is the spring force.

\(T_i\) is the joint moment of joint \(i\)

\(g\) is the gravitational acceleration.

Based on the model used here (see figure 1) the following relation applies to angles \(\alpha, \beta,\) and \(\theta:\)

\[\alpha_1 = \beta_1 = \theta_1\]

\[\alpha_2 = \theta_2 - \alpha_1 \quad \beta_2 = \pi - \alpha_2\]

\[1\]
\[ \alpha_3 = \theta_3 - \alpha_2 - \alpha_1 \quad \beta_3 = \pi - \alpha_3 \]

\( \alpha_i \) and \( \beta_i \) are the angles in the "joint space", \( \beta_i \) is the internal angle of the joint \( i \) and \( \alpha_i \) the external one; \( \theta_i \) is the angle of the joint \( i \) in the "segment space".

\[ \alpha_i \text{ and } \beta_i \text{ are the angles in the "joint space", } \beta_i \text{ is the internal angle of the joint } i \text{ and } \alpha_i \text{ the external one; } \theta_i \text{ is the angle of the joint } i \text{ in the "segment space".} \]

**Figure 1.** Model of the human body for the arm movement in the sagittal plane with the schematic location of the initial and target position. In this figure the spring force accounts for the external force on the hand (see the text).

In order to compute the equations of motion, the linear accelerations of the center of gravity of each link taking into account the constraints imposed by the kinematics of the linkage and starting from the shoulder joint as a fixed reference point were calculated by the first derivative of the Jacobian, \( J_i \), of the respective angular velocities; or in a formal matrix form:

\[ [\ddot{x}_i \ ¿ \ \ddot{y}_i]^T = \dot{J}_i [\dot{\alpha}_1 \ \dot{\alpha}_2 \ \dot{\alpha}_3]^T + J_i [\ddot{\alpha}_1 \ \ddot{\alpha}_2 \ \ddot{\alpha}_3]^T \quad i = 1...3 \]  \[ (2) \]

The linear accelerations of the segment's center of mass represented in figures 1 and 6 can be derived in a explicit form by:

**Upper arm:**

\[ x_1 = d_1 \cos \alpha_1 \]  \[ (3) \]

\[ \ddot{x}_1 = -d_1 (\ddot{\alpha}_1 \sin \alpha_1 + \dot{\alpha}_1^2 \cos \alpha_1) \]  \[ (4) \]

\[ y_1 = d_1 \sin \alpha_1 \]  \[ (5) \]
\[ y_1 = d_1 (\ddot{\alpha}_1 \cos \alpha_1 - \dot{\alpha}_1^2 \sin \alpha_1) \]  

(6)

**Forearm:**

\[ x_2 = \ell_1 \cos \alpha_1 + d_2 \cos(\alpha_1 + \alpha_2) \]  

(7)

\[ \dot{x}_2 = -[\ell_1 \sin \alpha_1 + d_2 \sin(\alpha_1 + \alpha_2)]\dot{\alpha}_1 - d_2 \sin(\alpha_1 + \alpha_2)\ddot{\alpha}_2 \]

\[-[\ell_1 \cos \alpha_1 + d_2 \cos(\alpha_1 + \alpha_2)]\dot{\alpha}_1^2 - d_2 \cos(\alpha_1 + \alpha_2)(2\dot{\alpha}_1 \dot{\alpha}_2 + \ddot{\alpha}_2^2) \]  

(8)

\[ y_2 = \ell_1 \sin \alpha_1 + d_2 \sin(\alpha_1 + \alpha_2) \]  

(9)

\[ \dot{y}_2 = [\ell_1 \cos \alpha_1 + d_2 \cos(\alpha_1 + \alpha_2)]\dot{\alpha}_1 + d_2 \cos(\alpha_1 + \alpha_2)\dot{\alpha}_2 \]

\[-[\ell_1 \sin \alpha_1 - d_2 \sin(\alpha_1 + \alpha_2)]\dot{\alpha}_1^2 + d_2 \sin(\alpha_1 + \alpha_2)(2\dot{\alpha}_1 \dot{\alpha}_2 + \ddot{\alpha}_2^2) \]  

(10)

**Hand:**

\[ x_3 = \ell_1 \cos \alpha_1 + \ell_2 \cos(\alpha_1 + \alpha_2) + d_3 \cos(\alpha_1 + \alpha_2 + \alpha_3) \]  

(11)

\[ \dot{x}_3 = -[\ell_1 \sin \alpha_1 + \ell_2 \sin(\alpha_1 + \alpha_2) + d_3 \sin(\alpha_1 + \alpha_2 + \alpha_3)]\dot{\alpha}_1 \]

\[-[\ell_2 \sin(\alpha_1 + \alpha_2) + d_3 \sin(\alpha_1 + \alpha_2 + \alpha_3)]\dot{\alpha}_2 - d_3 \sin(\alpha_1 + \alpha_2 + \alpha_3)\ddot{\alpha}_3 \]

\[-[\ell_1 \cos \alpha_1 + \ell_2 \cos(\alpha_1 + \alpha_2) + d_3 \cos(\alpha_1 + \alpha_2 + \alpha_3)]\dot{\alpha}_1^2 \]

\[-[\ell_2 \cos(\alpha_1 + \alpha_2) + d_3 \cos(\alpha_1 + \alpha_2 + \alpha_3)]\dot{\alpha}_2^2 - d_3 \cos(\alpha_1 + \alpha_2 + \alpha_3)\ddot{\alpha}_3^2 \]  

(12)

\[ y_3 = \ell_1 \sin \alpha_1 + \ell_2 \sin(\alpha_1 + \alpha_2) + d_3 \sin(\alpha_1 + \alpha_2 + \alpha_3) \]  

(13)

\[ \dot{y}_3 = [\ell_1 \cos \alpha_1 + \ell_2 \cos(\alpha_1 + \alpha_2) + d_3 \cos(\alpha_1 + \alpha_2 + \alpha_3)]\dot{\alpha}_1 \]

\[+[\ell_2 \cos(\alpha_1 + \alpha_2) + d_3 \cos(\alpha_1 + \alpha_2 + \alpha_3)]\dot{\alpha}_2 + d_3 \cos(\alpha_1 + \alpha_2 + \alpha_3)\ddot{\alpha}_3 \]

\[-[\ell_1 \sin \alpha_1 + \ell_2 \sin(\alpha_1 + \alpha_2) + d_3 \sin(\alpha_1 + \alpha_2 + \alpha_3)]\dot{\alpha}_1^2 \]

\[-[\ell_2 \sin(\alpha_1 + \alpha_2) + d_3 \sin(\alpha_1 + \alpha_2 + \alpha_3)]\dot{\alpha}_2^2 - d_3 \sin(\alpha_1 + \alpha_2 + \alpha_3)\ddot{\alpha}_3^2 \]  

(14)

\[ -[\ell_2 \sin(\alpha_1 + \alpha_2) + d_3 \cos(\alpha_1 + \alpha_2 + \alpha_3)]2\dot{\alpha}_1 \dot{\alpha}_2 \]

\[-d_3 \sin(\alpha_1 + \alpha_2 + \alpha_3)(2\dot{\alpha}_1 \dot{\alpha}_3 + 2\dot{\alpha}_2 \dot{\alpha}_3) \]

Based on the free body diagrams, the equations of motion in the sagittal plane were derived by means of the Newton-Euler method.

**Hand:**
\[ m_3 \ddot{x}_3 = F_{x3} - F_S \]  \hfill (15)

\[ m_3 \ddot{y}_3 = F_{y3} - m_3 g \]  \hfill (16)

\[ I_3 \ddot{\alpha}_3 = T_3 + F_{x3} d_3 \sin(\alpha_1 + \alpha_2 + \alpha_3) - F_{y3} d_3 \cos(\alpha_1 + \alpha_2 + \alpha_3) \]
\[ + F_3 d_3 \sin(\alpha_1 + \alpha_2 + \alpha_3) - I_3 \ddot{\alpha}_1 - I_3 \ddot{\alpha}_2 \]  \hfill (17)

**Forearm:**

\[ m_2 \ddot{x}_2 = F_{x2} - F_{x3} \]  \hfill (18)

\[ m_2 \ddot{y}_2 = F_{y2} - F_{y3} - m_2 g \]  \hfill (19)

\[ I_2 \ddot{\alpha}_2 = T_2 - T_3 + F_{x2} (\ell_2 - d_2) \sin(\alpha_1 + \alpha_2) - F_{y2} (\ell_2 - d_2) \cos(\alpha_1 + \alpha_2) \]
\[ + F_{x2} d_2 \sin(\alpha_1 + \alpha_2) - F_{y2} d_2 \cos(\alpha_1 + \alpha_2) - I_2 \ddot{\alpha}_1 \]  \hfill (20)

**Upper arm:**

\[ m_1 \ddot{x}_1 = F_{x1} - F_{x2} \]  \hfill (21)

\[ m_1 \ddot{y}_1 = F_{y1} - F_{y2} - m_1 g \]  \hfill (22)

\[ I_1 \ddot{\alpha}_1 = T_1 - T_2 + F_{x2} (\ell_1 - d_1) \sin \alpha_1 - F_{y2} (\ell_1 - d_1) \cos \alpha_1 \]
\[ + F_{x1} d_1 \sin \alpha_1 - F_{y1} d_1 \cos \alpha_1 \]  \hfill (23)

The joint moments can be straightforwardly found solving the set of equations (15-23) in a top-down way (first the wrist moment and so on) once the inertial parameters and the kinematics data are known (the accelerations in the set of equations (3-14) are found first). If the kinematics data are time-series, the equations are implemented in any programming package and the time-series of the joint moments are obtained. In this paper all data processing was implemented in Matlab. In a matrix-vector form (useful for visualizing the moment components, coupling and transfer effects) the joint moments can be represented as:

\[ T = M(\alpha)\ddot{\alpha} + v(\alpha, \dot{\alpha}) + G(\alpha) + T_{ext}(\alpha) \]  \hfill (24)

or
\[
\begin{bmatrix}
T_1 \\
T_2 \\
T_3
\end{bmatrix}
= \begin{bmatrix}
M(\alpha)_{1,1} & M(\alpha)_{1,2} & M(\alpha)_{1,3} \\
M(\alpha)_{2,1} & M(\alpha)_{2,2} & M(\alpha)_{2,3} \\
M(\alpha)_{3,1} & M(\alpha)_{3,2} & M(\alpha)_{3,3}
\end{bmatrix}
\begin{bmatrix}
\ddot{\alpha}_1 \\
\ddot{\alpha}_2 \\
\ddot{\alpha}_3
\end{bmatrix}
+ \begin{bmatrix}
\nu(\alpha, \dot{\alpha})_1 \\
\nu(\alpha, \dot{\alpha})_2 \\
\nu(\alpha, \dot{\alpha})_3
\end{bmatrix}
+ \begin{bmatrix}
G(\alpha)_1 \\
G(\alpha)_2 \\
G(\alpha)_3
\end{bmatrix}
+ \begin{bmatrix}
T_{\text{ext}}(\alpha)_1 \\
T_{\text{ext}}(\alpha)_2 \\
T_{\text{ext}}(\alpha)_3
\end{bmatrix}
\] (24a)

Where:

- \( T \) is the vector of joint moments (3x1).
- \( M(\alpha) \) is the inertia matrix (3x3). Because inertia matrices are symmetric (\( I_{1,2} = I_{2,1}, I_{1,3} = I_{3,1} \) and \( I_{2,3} = I_{3,2} \)), only six elements of the matrix should be determined. The elements are the three moments of inertia, \( I_{1,1}, I_{2,2} \) and \( I_{3,3} \).
- \( \ddot{\alpha} \) is the vector of angular accelerations (3x1).
- \( \nu(\alpha, \dot{\alpha}) \) is the vector of centrifugal/Coriolis terms (3x1).
- \( G(\alpha) \) is the vector of gravity terms (3x1).
- \( T_{\text{ext}} \) is the vector of joint moments due to other external forces besides gravity; in this case, it represents the moment due to the spring force (3x1).

The motion equations were rearranged in the above format and the correspondent terms are:

\[
T = \begin{bmatrix}
T_1 \\
T_2 \\
T_3
\end{bmatrix}^T
\]

\[
\ddot{\alpha} = \begin{bmatrix}
\ddot{\alpha}_1 \\
\ddot{\alpha}_2 \\
\ddot{\alpha}_3
\end{bmatrix}^T
\]

\[
M(\alpha)_{1,1} = m_1d_1^2 + I_1 + m_2(\ell_1^2 + d_2^2 + 2\ell_1d_2\cos\alpha_2) + I_2
+ m_3[\ell_1^2 + \ell_2^2 + d_3^2 + 2\ell_1\ell_2\cos\alpha_2 + 2\ell_1d_3\cos(\alpha_2 + \alpha_3) + 2\ell_2d_3\cos\alpha_3] + I_3
\] (27)

\[
M(\alpha)_{1,2} = m_2(d_2^2 + \ell_1d_2\cos\alpha_2) + I_2 + m_3[\ell_2^2 + d_3^2 + \ell_1\ell_2\cos\alpha_2
+ \ell_1d_3\cos(\alpha_2 + \alpha_3) + 2\ell_2d_3\cos\alpha_3] + I_3
\] (28)

\[
M(\alpha)_{1,3} = m_3[d_3^2 + \ell_1d_3\cos(\alpha_2 + \alpha_3) + \ell_2d_3\cos\alpha_3] + I_3
\] (29)

\[
M(\alpha)_{2,1} = m_2(d_2^2 + \ell_1d_2\cos\alpha_2) + I_2
+ m_3[\ell_2^2 + d_3^2 + \ell_1\ell_2\cos\alpha_2 + 2\ell_2d_3\cos\alpha_3 + \ell_1d_3\cos(\alpha_2 + \alpha_3)] + I_3
\] (30)
\[ M(\alpha)_{2,2} = m_2 d_2^2 + I_2 + m_3 (\ell_2^2 + d_3^2 + \ell_2 d_3 \cos \alpha_3) + I_3 \]  
(31)

\[ M(\alpha)_{2,3} = m_3 (d_3^2 + \ell_2 d_3 \cos \alpha_3) + I_3 \]  
(32)

\[ M(\alpha)_{3,1} = m_3 [d_3^2 + \ell_1 d_3 \cos (\alpha_2 + \alpha_3) + \ell_2 d_3 \cos \alpha_3] + I_3 \]  
(33)

\[ M(\alpha)_{3,2} = m_3 (d_3^2 + \ell_2 d_3 \cos \alpha_3) + I_3 \]  
(34)

\[ M(\alpha)_{3,3} = m_3 d_3^2 + I_3 \]  
(35)

\[ v(\alpha, \dot{\alpha})_1 = -[(m_2 \ell_1 d_2 + m_3 \ell_1 \ell_2) \sin \alpha_2 + m_3 \ell_1 d_3 \sin(\alpha_2 + \alpha_3)](2\dot{\alpha}_1 \dot{\alpha}_2 + \dot{\alpha}_2^2) 
- [m_3 \ell_2 d_3 \sin(\alpha_2 + \alpha_3) + m_3 \ell_2 d_3 \sin \alpha_3](2\dot{\alpha}_1 \dot{\alpha}_3 + 2\dot{\alpha}_2 \dot{\alpha}_3 + \dot{\alpha}_3^2) \]  
(36)

\[ v(\alpha, \dot{\alpha})_2 = [(m_3 \ell_1 \ell_2 + m_3 d_2 \ell_1) \sin \alpha_2 + m_3 d_3 \ell_1 \sin(\alpha_2 + \alpha_3)]\dot{\alpha}_1^2 
- m_3 d_3 \ell_2 \sin \alpha_3(2\dot{\alpha}_1 \dot{\alpha}_3 + 2\dot{\alpha}_2 \dot{\alpha}_3 + \dot{\alpha}_3^2) \]  
(37)

\[ v(\alpha, \dot{\alpha})_3 = [m_3 \ell_1 d_3 \sin(\alpha_2 + \alpha_3) + m_3 d_3 \ell_1 \sin \alpha_3] \dot{\alpha}_1^2 
+ m_3 \ell_2 d_3 \sin \alpha_3(2\dot{\alpha}_1 \dot{\alpha}_2 + \dot{\alpha}_2^2) \]  
(38)

\[ G(\alpha)_1 = m_1 g d_1 \cos \alpha_1 + m_2 g [\ell_1 \cos \alpha_1 + d_2 \cos(\alpha_1 + \alpha_2)] 
+ m_3 g [\ell_1 \cos \alpha_1 + \ell_2 \cos(\alpha_1 + \alpha_2) + d_3 \cos(\alpha_1 + \alpha_2 + \alpha_3)] \]  
(39)

\[ G(\alpha)_2 = m_2 g d_2 \cos(\alpha_1 + \alpha_2) + m_3 g [\ell_2 \cos(\alpha_1 + \alpha_2) + d_3 \cos(\alpha_1 + \alpha_2 + \alpha_3)] \]  
(40)

\[ G(\alpha)_3 = m_3 g d_3 \cos(\alpha_1 + \alpha_2 + \alpha_3) \]  
(41)

\[ T_{ext}(\alpha)_1 = -F_S \ell_1 \sin \alpha_1 - F_S \ell_2 \sin(\alpha_1 + \alpha_2) - F_S (d_3 + d_3) \sin(\alpha_1 + \alpha_2 + \alpha_3) \]  
(42)

\[ T_{ext}(\alpha)_2 = -F_S \ell_2 \sin(\alpha_1 + \alpha_2) - F_S (d_3 + d_3) \sin(\alpha_1 + \alpha_2 + \alpha_3) \]  
(43)

\[ T_{ext}(\alpha)_3 = -F_S (d_3 + d_3) \sin(\alpha_1 + \alpha_2 + \alpha_3) \]  
(44)